TAME COVERS AND COHOMOLOGY OF RELATIVE CURVES OVER COMPLETE DISCRETE VALUATION RINGS WITH APPLICATIONS TO THE BRAUER GROUP

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Abstract. We prove the existence of noncrossed product division algebras and indecomposable division algebras of unequal period and index over the function field of any p-adic curve, extending the results and methods of [10].

1. Introduction

We study the cohomology and the Brauer group of a field F that is finitely generated and of transcendence degree one over the p-adic field $\mathbb{Q}_p$. Such a field is always the function field of a regular (projective, flat) relative curve $X/\mathbb{Z}_p$. In [10] it was shown that if F admits a *smooth* model $X/\mathbb{Z}_p$ then there exist noncrossed product F-division algebras, and indecomposable F-division algebras of unequal prime-power period and index. These were constructed from objects defined over the generic point (p) of the closed fiber $X_0 = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ using a homomorphism $\lambda : \mathrm{Br}(F_p)' \to \mathrm{Br}(F)$ that splits the restriction map $\mathrm{res} : \mathrm{Br}(F) \to \mathrm{Br}(F_p)$, where the subscript p denotes completion with respect to the valuation defined by (p), and "$\,'\,$" denotes "prime-to-p part". The fields F that have a smooth $X/\mathbb{Z}_p$ include fields such as $\mathbb{Q}_p(t)$ but do not include the function fields of *all* p-adic curves.

In this paper we generalize the machinery and results of [10] to arbitrary p-adic curves. We prove that if F is the function field of a p-adic curve then there exist noncrossed product F-division algebras, and indecomposable F-division algebras of unequal prime-power period and index. The machinery we develop here is used in [11] to prove that every F-division algebra of prime period ℓ and index ℓ^2 decomposes into two cyclic F-tensor factors, hence is a crossed product, generalizing Suresh's result [36], which assumes roots of unity. In the terminology of [5, Sections 3,4], this shows the $\mathbb{Z}/\ell$-length in $\mathrm{H}^2(F, \mu_\ell)$ equals the ℓ-Brauer dimension, which is two by a theorem of Saltman ([33, Theorem 3.4]). In general our work is motivated by the work of Saltman over these fields in [33] and [35] (see also [9]).

We summarize the technical results. Let R be a complete discrete valuation ring with fraction field K, and let F be a finitely generated field extension of K of transcendence degree one. Let X/R be a regular (projective, flat) model for F whose closed fiber X_0 has normal crossings on X. Let $C = X_{0,\mathrm{red}}$, let $\{C_i\}$ denote the set of irreducible components of C, let $\mathcal{S}$ be the set of singular points of C, and set $F_C = \prod_i F_{C_i}$, the product of the completions with respect to the valuations

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on F defined by the C_i. We construct for any prime-to-char($\kappa(C)$) number n, any integer r, and any $q \geq 0$, a homomorphism

$$\lambda : \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r)) \to \mathrm{H}^q(\mathrm{O}_{X,\mathcal{S}}, \mathbb{Z}/n(r)) \to \mathrm{H}^q(F, \mathbb{Z}/n(r))$$

that splits the restriction map $\mathrm{H}^q(\mathrm{O}_{X,\mathcal{S}}, \mathbb{Z}/n(r)) \to \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r))$. We use λ to construct another map λ that splits the subgroup of the image of $\mathrm{H}^q(F, \mathbb{Z}/n(r)) \to \mathrm{H}^q(F_C, \mathbb{Z}/n(r))$ consisting of tuples of classes that are unramified at $\mathcal{S}$ and glue along $\mathcal{S}$. When $q = 2$, $\mathbb{Z}/n(r) = \mu_n$, and $R = \mathbb{Z}_p$, we show that λ preserves index. This allows us to construct indecomposable F-division algebras and noncrossed product F-division algebras as mentioned above, in the same manner as [10]. When the dual graph of the closed fiber X_0 has nontrivial topology, i.e., nonzero (first) Betti number, we construct cyclic covers of X that are (defined and) trivial at every point of X except the generic point of X. These arise as cyclic covers of the closed fiber X_0 that are trivial at every point, and transported to X via λ. When $R = \mathbb{Z}_p$ they are the completely split cyclic covers of Saito ([30]). We thank Colliot-Thélène for drawing our attention to these interesting specimens.

2. Background and Conventions

We use [26, Chapter 8,9], [23, Section 2], and [22, Chapter XIII] for many of the following definitions.

2.1. General Conventions. Let S be an excellent scheme, n a number that is invertible on S, and $\Lambda = (\mathbb{Z}/n)(i)$ the twisted étale sheaf. We write $\mathrm{H}^q(S, \Lambda)$ for the étale cohomology group, and if Λ is understood (or doesn't matter) we write $\mathrm{H}^q(S, r)$ instead of $\mathrm{H}^q(S, \Lambda(r))$, and $\mathrm{H}^q(S)$ in place of $\mathrm{H}^q(S, 0)$. If $S = \operatorname{Spec} A$ for a ring A then we write $\mathrm{H}^q(A, r)$. If T is an integral scheme contained in S we write $\kappa(T)$ for its function field. If $T \to S$ is a morphism of schemes then the restriction $\mathrm{res}_{T|S} : \mathrm{H}^q(T) \to \mathrm{H}^q(S)$ is defined, and we write $\beta_S = \mathrm{res}_{T|S}(\beta)$. If $Z \subset S$ is a subscheme, we write Z_T for the preimage $Z \times_S T$.

2.2. Valuation Theory. If v is a valuation on a field F we write $\kappa(v)$ for the residue field of the valuation ring O_v, and F_v for the completion of F at v. If S is a connected normal scheme with function field F and v arises from a prime divisor D on S, we write v_D for v, $\kappa(D)$ for $\kappa(v)$, and F_D for F_v. If D is a sum of prime divisors D_i we write $F_D = \prod_i F_{D_i}$. Each $f \in F^*$ defines a divisor $\mathrm{div}(f) = \sum v_D(f)D$, where the (finite) sum is over prime divisors on S.

Recall that if $F = (F, v)$ is a discretely valued field and $\alpha \in \mathrm{H}^q(F, \Lambda)$ then α has a *residue* $\partial_v(\alpha)$ in $\mathrm{H}^{q-1}(\kappa(v), \Lambda(-1))$. More generally, suppose T is a noetherian scheme, ξ is a generic point of T, and $\alpha \in \mathrm{H}^q(T, \Lambda)$. Then for each discrete valuation v on the field $F = \kappa(\xi_{\mathrm{red}})$ α has a residue

$$\partial_v(\alpha) \overset{\mathrm{df}}{=} \partial_v(\alpha_F) \in \mathrm{H}^{q-1}(\kappa(v), \Lambda(-1))$$

We say α is *unramified at* v if $\partial_v(\alpha) = 0$, *ramified at* v if $\partial_v(\alpha) \neq 0$, and *tamely ramified at* v if $\partial_v(\alpha)$ is contained in the prime-to-char($\kappa(v)$) part of $\mathrm{H}^{q-1}(\kappa(v), \Lambda(-1))$. If α is unramified at v the *value* of α at v is the element

$$\alpha(v) = \mathrm{res}_{F|F_v}(\alpha_F) \in \mathrm{H}^q(\kappa(v), \Lambda) \leq \mathrm{H}^q(F_v, \Lambda)$$

(see [16, 7.13, p.19]). Suppose $T \to S$ is a birational morphism of noetherian schemes (see [17, I.2.2.9]). The *ramification locus of* α *on* S_{red} is the sum of the

prime divisors on S_{red} that determine valuations at which α is ramified, over all generic points of S_{red}.

Let D be a divisor on a noetherian normal scheme S, set $U = S - D$, and for each generic point ξ of $\operatorname{Supp} D$, let K_ξ denote the fraction field of the discrete valuation ring $\mathrm{O}_{S,\xi}$. We say a morphism $\rho : T \to S$ is *tamely ramified along* D if T is normal, $\rho_U : V = T \times_S U \to U$ is étale, and for each generic point ξ of $\operatorname{Supp} D$, the étale K_ξ-algebra L defined by $\operatorname{Spec} L = V \times_U \operatorname{Spec} K_\xi$ is tamely ramified with respect to $\mathrm{O}_{S,\xi}$. Since L/K_ξ is étale it is a finite product of separable field extensions of K_ξ, and L is tamely ramified if each field extension is tamely ramified (with respect to $\mathrm{O}_{S,\xi}$) in the usual sense. If S is a noetherian scheme whose irreducible components are normal, we'll say a morphism $\rho : T \to S$ is tamely ramified along D if again $V \to U = S - D$ is étale, and the restriction ρ_{S_i} to each irreducible component S_i of S is tamely ramified along D_{S_i}. If $S = \operatorname{Spec} A$ and $T = \operatorname{Spec} B$, we will also say B is a *tamely ramified A-algebra*. We say a map $\rho : T \to S$ of noetherian schemes is a *cover* if it is finite, generically étale, and each connected component of T dominates a connected component of S.

2.3. Relative Curves. In this paper, R will be a complete discrete valuation ring with residue field k and fraction field K, F will be a field finitely generated of transcendence degree one over K, and X/R will be a regular 2-dimensional scheme X that is flat and projective over $\operatorname{Spec} R$ and has function field $K(X) = F$. We call X/R a *regular relative curve*, write $X_0 = X \otimes_R k$ for the closed fiber, $C = X_{0,\mathrm{red}}$ for the reduced subscheme underlying the closed fiber, and $C_1, \ldots, C_m$ for the irreducible components of C. We assume each C_i is regular, and at most two of them meet at any closed point of X, a situation that can always be obtained by blowing up using Lipman's embedded resolution theorem (see [26, 9.2.4]). For all closed points $z \in X$, we have $\dim \mathrm{O}_{X,z} = 2$ by [26, 8.3.4(c)], and since X is regular, $\mathrm{O}_{X,z}$ is factorial by Auslander-Buchsbaum's theorem.

We say an effective divisor D on a relative curve X/R is *horizontal* if each of its irreducible components maps surjectively (hence finitely) to $\operatorname{Spec} R$, and *vertical* if its support is contained in the support of the closed fiber. If D is a reduced and irreducible horizontal divisor then it is flat over $\operatorname{Spec} R$, since R is a discrete valuation ring. Every effective divisor on a relative curve X/R is a sum of horizontal and vertical divisors, and the horizontal prime divisors are exactly the closures of the closed points of the generic fiber ([26, 8.3.4(b)]). Since R is henselian, each irreducible horizontal divisor has a single closed point.

2.4. Distinguished Divisors. In general there will be many horizontal divisors on a relative curve X that restrict to a given divisor on C. In order to construct our lifts of covers and cohomology classes from C to X we select a single regular horizontal divisor for each closed point, as follows.

Proposition 2.5. *Assume the setup of* (2.3). *Then for each closed $z \in X$ there exists a regular irreducible horizontal divisor $D \subset X$ that intersects each irreducible component of C passing through z transversally at z.*

Proof. Transversal intersection with a single component is by [26, 8.3.35(g)] and its proof (see also [21, 21.9.12]). Thus if $z \in C_i \cap C_j$ $(i \neq j)$ and t_i and t_j are local equations for C_i and C_j, then we have local equations f_i and f_j for effective

regular horizontal divisors such that $(f_i, t_i) = (f_j, t_j) = \mathfrak{m}_z \subset \mathrm{O}_{X,z}$. If $(f_j, t_i) = \mathfrak{m}_z$ or $(f_i, t_j) = \mathfrak{m}_z$ then a suitable D is defined locally by f_j or f_i. Otherwise $(f_i + f_j, t_i) = (f_i + f_j, t_j) = \mathfrak{m}_z$, and we define D locally by $f_i + f_j$. The rest of the proof proceeds as in [26, 3.3.35]. □

We fix a set of these (prime) divisors, and let $\mathscr{D}$ denote the set of supports of the semigroup they generate in $\operatorname{Div} X$. We will say a divisor D is *distinguished* and write $D \in \mathscr{D}$ whenever D is reduced and supported in $\mathscr{D}$. Though $\mathscr{D}$ is fixed in principle, we reserve the right to declare any divisor satisfying the definition to be a member of this set retroactively. Let $\mathscr{D}_{\mathcal{S}}$ denote the subset that *avoids* $\mathcal{S}$. Note that each $D \in \mathscr{D}$ is a disjoint union of its irreducible components, each of which meets each irreducible component of C transversally.

3. Structure of Tame Covers

Lemma 3.1 (Structure)**.** *Assume the setup of (2.3). Suppose $\rho : Y \to (X, D)$ is a tamely ramified cover, where $D \in \mathscr{D}$. Then*

a) *The structure map $\rho : Y \to X$ is flat.*
b) *Y/R is a regular relative curve, $Y_{0,\mathrm{red}} = C_Y$, each irreducible component of C_Y is regular, $\mathcal{S}_Y$ is the set of singular closed points of C_Y, and exactly two irreducible components of C_Y meet at each point of $\mathcal{S}_Y$.*
c) *The support of the irreducible components of D'_Y for $D' \in \mathscr{D}$ generate a set $\mathscr{D}_Y$ of distinguished divisors on Y.*

Proof. Since $Y \to X$ is finite, $\dim(X) = \dim(Y) = 2$ by [19, 5.4.2], and $Y \to \operatorname{Spec} R$ is projective as the composition of projective morphisms ([26, 3.3.32]). Let $y \in Y$ be a closed point and set $x = f(y)$, $A = \mathrm{O}_{X,x}$, $B' = \mathrm{O}_{Y,x}$, and $B = \mathrm{O}_{Y,y}$. Choose a geometric point over x that lifts to each point of Y lying over x, and use this in the following to define the strict henselizations with respect to the maximal ideals of these points.

Since the statements involving D are local and D is a disjoint union of its irreducible components we may assume D is irreducible. Let $C_i \subset C$ be a (regular) irreducible component going through x, and let $\{f, t\} \subset A$ be the regular system of parameters formed by local equations for the distinguished prime divisor passing through x, and for C_i, respectively. Then the strict henselization A^{sh} of A with respect to the maximal ideal of A is a two-dimensional regular local ring, faithfully flat over A, with regular system of parameters $\{f, t\}$ (see [21, 18.8]).

If $x \notin D$ then $B' \otimes_A A^{\mathrm{sh}}$ is a finite étale A^{sh}-algebra by base change, since $\rho|_{X-D}$ is finite-étale. If $x \in D$ then $B' \otimes_A A^{\mathrm{sh}}$ is a finite tamely ramified A^{sh}-algebra by [23, Lemma 2.2.8]. By [21, 18.8.10], B^{sh} is a factor of the direct product decomposition of $B' \otimes_A A^{\mathrm{sh}}$, hence B^{sh} is a finite tamely ramified local A^{sh}-algebra, in particular it is a normal local ring, hence it is a normal domain. It follows that B^{sh} is the integral closure of A^{sh} in the field $\widetilde{L} \overset{\mathrm{df}}{=} \operatorname{Frac} B^{\mathrm{sh}}$. Since the tame fundamental group of the strictly henselian regular local ring A^{sh} is abelian ([22, XIII.5.3]) the field extension $\widetilde{L}/\operatorname{Frac} A^{\mathrm{sh}}$ is Galois, and by Abhyankar's Lemma ([13, A.I.11], see also [23, Corollary 2.3.4])

$$B^{\mathrm{sh}} = A^{\mathrm{sh}}[T]/(T^e - f) \quad (\text{some } e \geq 1)$$

By [23, Lemma 1.8.6] B^{sh} is a regular (2-dimensional) local ring with system of parameters $\{\sqrt[e]{f}, t\}$. Since $B \to B^{\mathrm{sh}}$ is faithfully flat and B^{sh} is regular, B is regular by flat descent ([19, 6.5.1] or [27, 23.7(i)]), and since B is the local ring of an arbitrary closed point, we conclude Y is regular. It follows that $\rho : Y \to X$ is flat by [27, 23.1], proving (a), and since Y is regular and $Y \to \operatorname{Spec} R$ is flat and projective, Y/R is a regular relative curve.

We derive a system of parameters for B. The prime ideal $(\sqrt[e]{f}) \subset B^{\mathrm{sh}}$ is the only one lying over $(f)A^{\mathrm{sh}}$ since, for $\kappa(f) = \operatorname{Frac} A^{\mathrm{sh}}/(f)A^{\mathrm{sh}}$, the ring $B^{\mathrm{sh}} \otimes_{A^{\mathrm{sh}}} \kappa(f) = \kappa(f)[T]/(T^e)$ of the fiber over $\operatorname{Spec} \kappa(f)$ consists of a single prime ideal. The image $(\sqrt[e]{f})$ in $\operatorname{Spec} B$ is therefore a unique prime $(g) \subset B$ lying over $(f) \subset A$, and $(\sqrt[e]{f})$ is the unique prime lying over (g). Therefore, since $B \to B^{\mathrm{sh}}$ is unramified, $(g)B^{\mathrm{sh}} = (\sqrt[e]{f})$. Since $B \to B^{\mathrm{sh}}$ is faithfully flat, $IB^{\mathrm{sh}} \cap B = I$ for all ideals I of B (by e.g. [4, Exercise 3.16]), so since $(g,t)B^{\mathrm{sh}} = (\sqrt[e]{f}, t)$ is maximal, $(g,t)B^{\mathrm{sh}} \cap B = (g,t)$ is the maximal ideal of B. Thus $\{g, t\}$ is a regular system for B.

Since t is a local equation for $\rho^{-1}C_i$, $\rho^{-1}C_i$ is regular and irreducible at y for each C_i passing through x. In particular $C_Y = \bigcup_i \rho^{-1}C_i$ is reduced, and so equals $Y_{0,\mathrm{red}}$. Since at most two irreducible components of C meet at x, the same holds for C_Y at y, and y is a singular point on C_Y if and only if $x = f(y) \in \mathcal{S}$. This completes the proof of (b).

If $D' \in \mathscr{D}$ is the distinguished (horizontal) prime divisor running through x then there is a single irreducible component of D'_Y passing through y, whose support $D'_{Y,\mathrm{red}}$ has local equation g at y. Thus each irreducible component of D'_Y covers D', hence $\operatorname{Spec} R$, hence D'_Y is horizontal. Since g is part of the regular system $\{g, t\}$ at the arbitrary closed point y we see that $D'_{Y,\mathrm{red}}$ is regular, and since t is a local equation for an arbitrary irreducible component of C_Y passing through y, $D'_{Y,\mathrm{red}}$ intersects each component of C_Y transversally. Thus the support of the irreducible components of D'_Y generate a set of distinguished divisors $\mathscr{D}_Y$ for Y. This proves (c).

■

Lemma 3.2. *Suppose X is a regular noetherian scheme and L is an étale $K(X)$-algebra that is tamely ramified along a divisor D. Then the normalization Y of X in L defines a tamely ramified cover $\rho : Y \to (X, D)$.*

Proof. Since X is regular, its connected components are integral regular schemes, hence we may assume X is integral. Since $L/K(X)$ is étale, L is a product of finite separable field extensions of $K(X)$, hence we may assume $L/K(X)$ is itself a finite separable field extension. Then the normalization Y exists, Y is normal by definition, and $\rho : Y \to X$ is finite by [26, 4.1.25]. Since Y is normal and connected it is irreducible, so Y dominates X. Let $U = X - D$, and set $V = Y \times_X U$. Since X is normal, Y is connected, and $\rho|_V$ is unramified, $\rho|_V$ is étale by [22, I.9.11] (see also [28, I.3.20]). Therefore $Y \to (X, D)$ is a tamely ramified cover.

■

The next lemma shows how distinguished divisors split in tamely ramified covers.

Lemma 3.3. *Assume the setup of (2.3). Suppose $\rho : Y \to (X, D)$ is a tamely ramified cover, where $D \in \mathscr{D}$, and $D' \in \mathscr{D}_{\mathcal{S}}$ is irreducible. Suppose $E \subset D'_{Y,\mathrm{red}}$ is a distinguished prime divisor lying over D' as in Lemma 3.1(c), $y = E \times_Y C_Y$, and*

$x = D' \times_X C$. Then y and x are regular closed points, and the ramification (resp. inertia) degree of v_E over $v_{D'}$ equals the ramification (resp. inertia) degree of v_y over v_x.

Proof. Since we assume (2.3) and $D \in \mathscr{D}$ we have Lemma 3.1, which shows C_Y is reduced and $E \subset D'_{Y,\mathrm{red}}$ is distinguished. Note that either $D' \cap D = \varnothing$ or $D' \subset D$. Since D' and E are distinguished and avoid the singular points of X and Y, they intersect the reduced closed fibers C and C_Y transversally, hence $x = D' \times_X C$ and $y = E \times_Y C_Y$ are regular closed points. We must show that $[\kappa(E) : \kappa(D')] = [\kappa(y) : \kappa(x)]$ and that $v_E(f) = v_y(f_0)$, where $f \in \mathrm{O}_{X,D'}$ is a local equation for D' on X and $f_0 \in \mathrm{O}_{C,x}$ is a local equation for x on C.

Since D' is horizontal and irreducible, $D' = \operatorname{Spec} S$ for S a finite local R-algebra by [28, I.4.2], and S is a discrete valuation ring since D' is regular. The map $E \to \rho^{-1}D' \to D'$ is finite as a composition of finite morphisms, hence $E = \operatorname{Spec} T$ for T a finite local S-algebra, again a discrete valuation ring since E is regular. Since S is a discrete valuation ring and $S \to T$ is finite, T is a free S-module of finite rank, and so $[T : S]$ is well defined. Since the generic point of E lies over that of D', we have $\operatorname{Frac} T = T \otimes_S \operatorname{Frac} S$, hence $[\kappa(E) : \kappa(D')] = [T : S]$.

Let $A = \mathrm{O}_{X,x}$, $B = \mathrm{O}_{Y,y}$, let t be a local equation for C at x, and set $A_0 = A/(t)$ and $B_0 = B/(t)B$, the (reduced) local rings of the fibers through x and y, as in the proof of Lemma 3.1. Already $\kappa(x) = S \otimes_A A_0$ and $\kappa(y) = T \otimes_B B_0$ by the transversality of the intersections. Since $B_0 = B \otimes_A A_0$ we have $\kappa(y) = T \otimes_A A_0$, hence $[\kappa(y) : \kappa(x)] = [T : S] = [\kappa(E) : \kappa(D')]$ by base change.

Let $g \in A$ be defined as above. To compute the ramification degree, note that since $B \to B^{\mathrm{sh}}$ is faithfully flat, $(g^e)B = (g^e)B^{\mathrm{sh}} \cap B = (f)B^{\mathrm{sh}} \cap B = (f)B$, hence $g^e = fu$ for some $u \in B^*$. Since f and g are uniformizers for $v_{D'}$ and v_E, respectively, it follows that $e(v_E/v_{D'}) = v_E(f) = e$. On the other hand, let f_0 be the image of f in A_0, and let g_0 be the image of g in B_0. Then f_0 cuts out the closed point x on C and g_0 cuts out y on C_Y by transversality. Thus f_0 and g_0 are uniformizers for v_x and v_y, and since $g_0^e = f_0 u_0$, where u_0 is the image of u in B_0^*, we have $e(v_y/v_x) = v_y(f_0) = e$, as desired. This completes the proof. ∎

4. Lifting Cohomology Classes

4.1. Let k be a field, and let C/k be a reduced connected projective curve with regular irreducible components $C_1, \ldots, C_m$, at most two of which meet at any closed point. Denote the singular points of C by $\mathcal{S}$ and write $\mathrm{O}_{C,\mathcal{S}}$ for the semilocal ring $\varinjlim_U \mathrm{O}_C(U)$, where U varies over (dense) open subsets of C containing $\mathcal{S}$. Then $\mathrm{O}_{C,\mathcal{S}}$ is a subring of the rational function ring $\kappa(C) = \prod_i \kappa(C_i)$. For each $z \in \mathcal{S} \cap C_i$, let $K_{i,z} = \operatorname{Frac} \mathrm{O}^{\mathrm{h}}_{C_i,z}$, a field since z is a normal point of C_i, and if $\alpha_i \in \mathrm{H}^q(\kappa(C_i))$, let $\alpha_{i,z}$ denote the image in $\mathrm{H}^q(K_{i,z})$.

Lemma 4.2 (Gluing)**.** *Assume the setup of (4.1). There exists an element $\alpha \in \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}, \Lambda)$ that restricts to $\alpha_C = (\alpha_1, \ldots, \alpha_m) \in \bigoplus_i \mathrm{H}^q(\kappa(C_i), \Lambda)$ if and only if α_i is unramified at each $z \in \mathcal{S} \cap C_i$, and $\alpha_{i,z} = \alpha_{j,z}$ whenever $z \in C_i \cap C_j$.*

Proof. There is an exact sequence ([28, III.1.25])

$$
\begin{aligned}
0 \longrightarrow \mathrm{H}^0_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^0(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^0(\kappa(C)) \longrightarrow \mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow& \\
\longrightarrow \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^1(\kappa(C)) \longrightarrow \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow& \\
\longrightarrow \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^2(\kappa(C)) \longrightarrow \mathrm{H}^3_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}})&
\end{aligned}
\tag{$*$}
$$

where the maps into the direct sum are restrictions. Since $\mathcal{S}$ is a disjoint union of closed points, $\mathrm{H}^q_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}) = \bigoplus_{z\in\mathcal{S}} \mathrm{H}^q_z(\mathrm{O}_{C,\mathcal{S}}) = \bigoplus_{z\in\mathcal{S}} \mathrm{H}^q_z(\mathrm{O}^{\mathrm{h}}_{C,z})$ by excision ([28, III.1.28, p.93]). Since Λ is a smooth group scheme, $\mathrm{H}^q(\mathrm{O}^{\mathrm{h}}_{C,z}) = \mathrm{H}^q(\kappa(z))$, by the cohomological Hensel's lemma [28, III.3.11(a), p.116]. Since the C_k are regular and at most two of them meet at any $z \in \mathcal{S}$, we have $\operatorname{Spec} \mathrm{O}^{\mathrm{h}}_{C,z} - \{z\} = \operatorname{Spec}(K_{i,z} \times K_{j,z})$ for some i and j, and an "excised" exact sequence

$$
\begin{aligned}
0 \longrightarrow \mathrm{H}^0_z(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^0(\kappa(z)) \longrightarrow \mathrm{H}^0(K_{i,z} \times K_{j,z}) \longrightarrow \mathrm{H}^1_z(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow& \\
\longrightarrow \mathrm{H}^1(\kappa(z)) \longrightarrow \mathrm{H}^1(K_{i,z} \times K_{j,z}) \longrightarrow \mathrm{H}^2_z(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow& \\
\longrightarrow \mathrm{H}^2(\kappa(z)) \longrightarrow \mathrm{H}^2(K_{i,z} \times K_{j,z}) \longrightarrow \mathrm{H}^3_z(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \cdots&
\end{aligned}
$$

where the map $\mathrm{H}^q(\kappa(z)) \to \mathrm{H}^q(K_{i,z} \times K_{j,z}) = \mathrm{H}^q(K_{i,z}) \oplus \mathrm{H}^q(K_{j,z})$ is the diagonal map given by inflation from $\kappa(z)$ to the "local fields" $K_{i,z}$ and $K_{j,z}$. Since n is prime-to-p, the map $\mathrm{H}^0(\kappa(z)) \to \mathrm{H}^0(K_{i,z})$ is an isomorphism, so $\mathrm{H}^0_z(\mathrm{O}_{C,\mathcal{S}}) = 0$, and for $q \geq 1$ we have short exact Witt-type sequences

$$0 \to \mathrm{H}^q(\kappa(z)) \to \mathrm{H}^q(K_{i,z}) \xrightarrow{\partial_z} \mathrm{H}^{q-1}(\kappa(z), -1) \to 0$$

Thus the long exact sequence breaks up into short exact sequences

$$0 \to \mathrm{H}^q(\kappa(z)) \to \mathrm{H}^q(K_{i,z} \times K_{j,z}) \to \mathrm{H}^{q+1}_z(\mathrm{O}_{C,\mathcal{S}}) \to 0 \qquad (q \geq 0) \tag{4.3}$$

By the compatibility of the localization sequence with the excised sequence, the map $\mathrm{H}^q(\kappa(C_i)) \to \mathrm{H}^{q+1}_z(\mathrm{O}_{C,\mathcal{S}}) \leq \mathrm{H}^{q+1}_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}})$ of $(*)$ factors through $\mathrm{res}_{\kappa(C_i)|K_{i,z}}$. Therefore an element $\alpha_C = (\alpha_1, \ldots, \alpha_m) \in \mathrm{H}^q(\kappa(C))$ maps to zero in $\mathrm{H}^{q+1}_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}})$ if and only if each couple $(\alpha_{i,z}, \alpha_{j,z})$ is in the image of some $\bar{\alpha} \in \mathrm{H}^q(\kappa(z))$; i.e., $\alpha_{i,z} = \alpha_{j,z}$, and both are unramified. Thus by the exactness of $(*)$, α_C comes from $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}})$ if and only if each α_i is unramified at each $z \in \mathcal{S} \cap C_i$, and $\alpha_{i,z} = \alpha_{j,z}$ whenever $z \in C_i \cap C_j$.

■

Suppose C is as in (4.1). Since exactly two irreducible components meet at any $z \in \mathcal{S}$ the *dual graph* G_C is defined, and consists of a vertex for each irreducible component of C and an edge for each singular point, such that an edge and a vertex are incident when the corresponding singular point lies on the corresponding irreducible component ([30, 2.23], see also [26, 10.1.48]). The (first) Betti number for G_C is $\beta_C \overset{\mathrm{df}}{=} \mathrm{rk}(\mathrm{H}_1(G_C, \mathbb{Z})) = N + E - V$, where V, E and N are the numbers of vertices, edges, and connected components of G_C, respectively.

Lemma 4.4. *Assume the setup of (4.1). Then:*

a) *For any integer r, $\mathrm{H}^1(C, \mathbb{Z}/n(r)) \to \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n(r))$ is injective.*

b) *The map* $\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n(q-1)) \to \mathrm{H}^q(\kappa(C),\mathbb{Z}/n(q-1))$ *is injective for* $q=0,2$*, and for* $q=1$ *we have*

$$\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n) \;\simeq\; (\mathbb{Z}/n)^{\beta_C}\oplus\Gamma$$

where $(\mathbb{Z}/n)^{\beta_C}$ *is the kernel of* $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)\to\mathrm{H}^1(\kappa(C),\mathbb{Z}/n)$*, and* $\Gamma\le \mathrm{H}^1(\kappa(C),\mathbb{Z}/n)$ *is the group of tuples that glue as in Lemma 4.2.*

Proof. We suppress the notation for $\Lambda=\mathbb{Z}/n(r)$. Let $z\in C-\mathcal{S}$ be a (regular) closed point, and set $U=C-\{z\}$, a dense open subset containing $\mathcal{S}$. The localization exact sequence is

$$\begin{aligned}0\longrightarrow \mathrm{H}^0_z(C)\longrightarrow&\mathrm{H}^0(C)\longrightarrow \mathrm{H}^0(U)\longrightarrow\cdots\\ &\cdots\longrightarrow \mathrm{H}^q_z(C)\longrightarrow \mathrm{H}^q(C)\longrightarrow \mathrm{H}^q(U)\longrightarrow \mathrm{H}^{q+1}_z(C)\longrightarrow\cdots\end{aligned}$$

By excision we have an exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{H}^0_z(C)\longrightarrow&\mathrm{H}^0(\mathrm{O}^{\mathrm{h}}_{C,z})\longrightarrow \mathrm{H}^0(K_z)\longrightarrow\cdots\\ &\cdots\longrightarrow \mathrm{H}^q_z(C)\longrightarrow \mathrm{H}^q(\mathrm{O}^{\mathrm{h}}_{C,z})\longrightarrow \mathrm{H}^q(K_z)\longrightarrow \mathrm{H}^{q+1}_z(C)\longrightarrow\cdots\end{aligned}$$

where $K_z=\operatorname{Frac}\mathrm{O}^{\mathrm{h}}_{C,z}$. Since z is a regular point $\mathrm{O}^{\mathrm{h}}_{C,z}$ is a discrete valuation ring, and by [12, Section 3.6] we may replace $\mathrm{H}^{q+1}_z(C)$ with $\mathrm{H}^{q-1}(\kappa(z),-1)$, and the map from $\mathrm{H}^q(U)$, which factors through $\mathrm{H}^q(K_z)$, is then the residue map ∂_z. We conclude $\mathrm{H}^0(C)=\mathrm{H}^0(U)$, and we have a long exact sequence

$$\begin{aligned}0\longrightarrow&\mathrm{H}^1(C)\longrightarrow \mathrm{H}^1(U)\xrightarrow{\partial_z}\mathrm{H}^0(\kappa(z),-1)\longrightarrow\cdots\\ &\cdots\longrightarrow \mathrm{H}^q(C)\longrightarrow \mathrm{H}^q(U)\xrightarrow{\partial_z}\mathrm{H}^{q-1}(\kappa(z),-1)\longrightarrow\cdots\end{aligned}\tag{$**$}$$

As $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}})=\varinjlim_U \mathrm{H}^1(U)$, where the limit is over dense open subsets of C containing $\mathcal{S}$, $\mathrm{H}^1(C)\to\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}})$ is injective by the exactness of the injective limit functor, proving (a).

For (b) we go back to $\Lambda=\mathbb{Z}/n$. By $(*)$ we have an exact sequence

$$\begin{aligned}0\longrightarrow \mathrm{H}^0(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)\xrightarrow{\phi_1}\mathrm{H}^0(\kappa(C),\mathbb{Z}/n)\xrightarrow{\phi_2}\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)\xrightarrow{\phi_3}\\ \xrightarrow{\phi_3}\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)\xrightarrow{\phi_4}\mathrm{H}^1(\kappa(C),\mathbb{Z}/n)\end{aligned}$$

The groups $\mathrm{H}^0(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)$ and $\mathrm{H}^0(\kappa(C),\mathbb{Z}/n)$ are finite free $\mathbb{Z}/n$-modules whose ranks are the number of C's connected components N and irreducible components m, respectively. We claim $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module. For by (4.3), for each $z\in\mathcal{S}$ we have an exact sequence

$$0\longrightarrow \mathrm{H}^0(\kappa(z),\mathbb{Z}/n)\longrightarrow \mathrm{H}^0(K_{i,z},\mathbb{Z}/n)\oplus\mathrm{H}^0(K_{j,z},\mathbb{Z}/n)\longrightarrow \mathrm{H}^1_z(\mathrm{O}^{\mathrm{h}}_{C,z},\mathbb{Z}/n)\to 0$$

This shows $\mathrm{H}^1_z(\mathrm{O}^{\mathrm{h}}_{C,z},\mathbb{Z}/n)\simeq\mathbb{Z}/n$, and since $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)$ is a finite direct sum of these groups, it is a finite free $\mathbb{Z}/n$-module, of rank $|\mathcal{S}|$.

The result [14, 27.1] implies that a free $\mathbb{Z}/n$-submodule of a $\mathbb{Z}/n$-module is a direct summand. Therefore we have a decomposition

$$\mathrm{H}^0(\kappa(C),\mathbb{Z}/n)\simeq\operatorname{im}(\phi_1)\oplus\operatorname{im}(\phi_2)$$

and since $\mathrm{H}^0(\kappa(C),\mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module, $\operatorname{im}(\phi_2)$ is a finite free $\mathbb{Z}/n$-module by the structure theorem for finitely generated abelian groups. Similarly,

since $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is a finite free $\mathbb{Z}/n$-module,

$$\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \simeq \mathrm{im}(\phi_2) \oplus \mathrm{cok}(\phi_2)$$

and since $\mathrm{im}(\phi_2)$ and $\mathrm{H}^1_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ are finite free $\mathbb{Z}/n$-modules, so is $\mathrm{cok}(\phi_2)$. Since $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ is a $\mathbb{Z}/n$-module, $\mathrm{cok}(\phi_2)$ is a direct summand of $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$, again by [14, 27.1]. Thus we have a decomposition

$$\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n) \simeq \mathrm{cok}(\phi_2) \oplus \mathrm{im}(\phi_4)$$

Now we set $\Gamma = \mathrm{im}(\phi_4)$, and compute $\mathrm{rk}(\mathrm{cok}(\phi_2)) = N + |\mathcal{S}| - m = \beta_C$. This proves the $q = 1$ part of (b).

The $q = 0$ case of (b) is in the proof of Lemma 4.2. Suppose $q = 2$. To show $\mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}, \mu_n) \to \mathrm{H}^2(\kappa(C), \mu_n)$ is injective, we will show $\mathrm{H}^1(\kappa(C), \mu_n) \to \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mu_n)$ is onto and apply the exactness of $(*)$.

For each closed point $z \in C_1 \cap C_2 \subset \mathcal{S}$, we have a diagram

$$\begin{tikzcd}
 & \mathrm{H}^1(\kappa(C_1), \mu_n) \oplus \mathrm{H}^1(\kappa(C_2), \mu_n) \arrow[r] \arrow[d] & \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n) \arrow[d, equal] & \\
0 \arrow[r] & \mathrm{H}^1(\kappa(z), \mu_n) \arrow[r] & \mathrm{H}^1(\kappa(C_1)_z, \mu_n) \oplus \mathrm{H}^1(\kappa(C_2)_z, \mu_n) \arrow[r] & \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n) \arrow[r] & 0
\end{tikzcd}$$

We will show that $\mathrm{H}^1(\kappa(C_1), \mu_n) \oplus \mathrm{H}^1(\kappa(C_2), \mu_n) \to \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n)$ is onto, by showing the downarrow is onto. Since z is a regular point of each C_i, each $\mathrm{O}_{C_i,z}$ is a discrete valuation ring with residue field $\kappa(z)$ and fraction field $\kappa(C_i)$, and we have a diagram of split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(\mathrm{O}_{C_i,z}, \mu_n) \arrow[r] \arrow[d] & \mathrm{H}^1(\kappa(C_i), \mu_n) \arrow[r] \arrow[d] & \mathrm{H}^0(\kappa(z), \mathbb{Z}/n) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{H}^1(\widehat{\mathrm{O}}_{C_i,z}, \mu_n) \arrow[r] & \mathrm{H}^1(\kappa(C_i)_z, \mu_n) \arrow[r] & \mathrm{H}^0(\kappa(z), \mathbb{Z}/n) \arrow[r] & 0
\end{tikzcd}$$

To show the middle downarrow is onto it suffices (by a standard diagram chase) to prove that the left downarrow is onto. Since $\widehat{\mathrm{O}}_{C_i,z}$ is henselian $\mathrm{H}^1(\widehat{\mathrm{O}}_{C_i,z}, \mu_n) = \mathrm{H}^1(\kappa(z), \mu_n)$, and by Kummer theory and Hilbert 90 we have $\mathrm{H}^1(\mathrm{O}_{C_i,z}, \mu_n) = \mathrm{O}^*_{C_i,z}/n$ and $\mathrm{H}^1(\kappa(z), \mu_n) = \kappa(z)^*/n$. Since $\mathrm{O}_{C_i,z} \to \kappa(z)$ is onto and $\mathrm{O}_{C_i,z}$ is local, the induced map $\mathrm{O}^*_{C_i,z} \to \kappa(z)^*$ is onto, hence $\mathrm{H}^1(\mathrm{O}_{C_i,z}, \mu_n)$ maps onto $\mathrm{H}^1(\kappa(z), \mu_n)$. We conclude $\mathrm{H}^1(\kappa(C_i), \mu_n) \to \mathrm{H}^1(\kappa(C_i)_z, \mu_n)$ is onto. Now each map $\mathrm{H}^1(\kappa(C_1), \mu_n) \oplus \mathrm{H}^1(\kappa(C_2), \mu_n) \to \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n)$ is onto.

Suppose $(b_z) \in \mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}}, \mu_n) = \bigoplus_{\mathcal{S}} \mathrm{H}^2_z(\mathrm{O}^{\mathrm{h}}_{C,z}, \mu_n)$. We have just seen that for each closed point $z \in C_i \cap C_j$ there exists a pair $([a_{i,z}t^{e_i}_{i,z}], [a_{j,z}t^{e_j}_{j,z}]) \in \kappa(C_i)^*/n \oplus \kappa(C_j)^*/n$ mapping to b_z, for z-units $a_{k,z} \in \mathrm{O}^*_{C_k,z}$, z-uniformizers $t_{k,z} \in \kappa(C_k)$, and integers e_k, for $k = i, j$. Let $v_{k,z}$ be the discrete valuation on $\kappa(C_k)$ determined by z. By standard approximation (e.g. [25, XII.1.2]) there exist elements $a_k, t_k \in \kappa(C_k)$ such that

$$v_{k,z}(a_k - a_{k,z}) > 0 \quad \text{and} \quad v_{k,z}(t_k - t_{k,z}) > 1 \quad \text{for all } z.$$

The image of $a_k t_k^{e_k}$ in $\kappa(C_k)^*_z/n$ is $[a_{k,z}t^{e_k}_{k,z}]$. Therefore the m-tuple

$$([a_k t_k^{e_k}]) \in \mathrm{H}^1(\kappa(C), \mu_n)$$

maps to (b_z). This proves the induced map $\mathrm{H}^1(\kappa(C),\mu_n)\to\mathrm{H}^2_{\mathcal{S}}(\mathrm{O}_{C,\mathcal{S}},\mu_n)$ is onto, and completes the proof.

■

We will soon need the following technical lemma in order to replace X_0 with C.

Lemma 4.5. *Suppose A is a noetherian ring. Then $(\operatorname{Frac}A)_{\mathrm{red}}=\operatorname{Frac}(A_{\mathrm{red}})$ if and only if A has no embedded primes.*

Proof. By definition $\operatorname{Frac}A=S^{-1}A$, where $S=A-\bigcup_{\operatorname{Ass}A}\mathfrak{p}$, and $S^{-1}N_A=N_{S^{-1}A}$ by [4, 3.12], hence $S^{-1}(A_{\mathrm{red}})=(S^{-1}A)_{\mathrm{red}}$. It remains to show that $S^{-1}(A_{\mathrm{red}})=\operatorname{Frac}(A_{\mathrm{red}})$ if and only if A has no embedded primes. By $S^{-1}(A_{\mathrm{red}})$ of course we mean $f(S)^{-1}(A_{\mathrm{red}})$, where $f:A\to A_{\mathrm{red}}$. This localization equals the localization with respect to the multiplicative set T, where T is the saturation of $f(S)$ in A_{red}, and this is the complement of the union of prime ideals of A_{red} that don't meet $f(S)$ by [4, Exercise 3.7]. Thus $S^{-1}(A_{\mathrm{red}})=\operatorname{Frac}(A_{\mathrm{red}})$ if and only if the union of the primes of A_{red} that don't meet $f(S)$ equals the union of the associated primes of A_{red}, which are just the minimal primes of A_{red}. But A and A_{red} have identical underlying topological spaces, and the primes of A_{red} that don't meet $f(S)$ correspond to the primes of A that don't meet S, i.e., the associated primes. These correspond to the minimal primes of A_{red} if and only if the associated primes of A are the minimal primes of A, i.e., A has no embedded primes. □

Theorem 4.6. *Assume the setup of (2.3), with X connected. Then for $q\ge 0$ there is a map*

$$\lambda:\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)\to\mathrm{H}^q(K(X),\Lambda)$$

and a commutative diagram

$$\begin{tikzcd} \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda) \arrow[r, "\lambda"] \arrow[d, "\oplus\mathrm{res}_i"'] & \mathrm{H}^q(K(X),\Lambda) \arrow[d, "\oplus_i\mathrm{res}_i"] \\ \mathrm{H}^q(\kappa(C),\Lambda) \arrow[r, "\inf"] & \bigoplus_i \mathrm{H}^q(K(X)_{C_i},\Lambda) \end{tikzcd} \tag{4.7}$$

such that if $\alpha_0\in\mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)$ and $\alpha=\lambda(\alpha_0)$ then:

a) *α is defined at the generic points of C_i, and $\alpha(C_i)=\mathrm{res}_i(\alpha_0)$.*
b) *The ramification locus of α (on X) is contained in $\mathscr{D}_{\mathcal{S}}$.*
c) *If $D\in\mathscr{D}_{\mathcal{S}}$ is prime and $z=D\cap C$, then $\partial_D\cdot\lambda=\inf_{\kappa(z)|\kappa(D)}\cdot\,\partial_z$.*
d) *If α_0 is unramified at a closed point z, and D is any (horizontal) prime lying over z, then α is unramified at D, and has value $\alpha(D)=\inf_{\kappa(z)|\kappa(D)}(\alpha_0(z))$.*

Proof. Let D_0 be an effective divisor on C that avoids $\mathcal{S}$, let $D\in\mathscr{D}_{\mathcal{S}}$ be the distinguished lift of D_0, set $U=X-D$, and set $U_0=C-D_0$. Since X and D are regular and D has pure codimension 1, we have $\mathrm{H}^0(X)\simeq\mathrm{H}^0(U)$, and an exact Gysin sequence

$$0\longrightarrow\mathrm{H}^1(X)\longrightarrow\mathrm{H}^1(U)\xrightarrow{\partial_D}\mathrm{H}^0(D,-1)\longrightarrow\mathrm{H}^2(X)\longrightarrow\cdots$$

by Gabber's absolute purity theorem ([15, Theorem 2.1.1]) and the standard construction of the Gysin sequence ([12, Section 3.2]). (Note that the result in [15] is stated for the $\Lambda=\mathbb{Z}/n$ case only, but the result holds in general since the sheaves $\mathscr{H}^q_D(X)$ and $\mathscr{H}^q_D(X,\mathbb{Z}/n)$ are locally isomorphic, and the morphism $i^*\Lambda(-1)\to$

$\mathscr{H}^2_D(X)$ is canonical.) We use the notation ∂_D since this map is compatible with the one defined above on $\mathrm{H}^q(K(X))$ when D is prime.

We may replace X_0 by $C = X_{0,\mathrm{red}}$ in the cohomological computations below since Λ is finite and n is prime-to-p, by [28, V.2.4(c)] (see also [28, II.3.11]). To substitute $\mathrm{O}_{C,\mathcal{S}}$ and $\kappa(C)$ for $\mathrm{O}_{X_0,\mathcal{S}}$ and $\kappa(X_0)$ we must check that the former are the canonical reduced quotients of the latter. But the ring $\mathrm{O}_{X_0,\mathcal{S}}$ can by obtained by localizing some affine open subset $\operatorname{Spec} A_0$ containing $\mathcal{S}$ (which exists since X_0/k is projective) with respect to the multiplicative set $T = A_0 - \bigcup_{\mathcal{S}} \mathfrak{m}_z$. Since $\mathrm{O}_{C,\mathcal{S}}$ is obtained by localizing $A_{0,\mathrm{red}}$ with respect to the image of T in $A_{0,\mathrm{red}}$, we have $\mathrm{O}_{C,\mathcal{S}} = (\mathrm{O}_{X_0,\mathcal{S}})_{\mathrm{red}}$ since the formation of the nilradical commutes with localization (see e.g. [4, 3.12]).

To show $\kappa(C) = \kappa(X_0)_{\mathrm{red}}$ it suffices to show X_0 has no embedded points by Lemma 4.5. But if z is any closed point of X then $\mathrm{O}_{X,z}$ is a regular local ring, and a local equation for the closed fiber $\mathrm{O}_{X,z} \otimes_R k$ passing through z is given by the uniformizer p in R. Since $\mathrm{O}_{X,z}$ is factorial and at most two components of X_0 pass through z we have $p = \pi_1^{e_1}\pi_2^{e_2}$ for primes π_i and numbers $e_i \geq 0$. The associated primes of $\mathrm{O}_{X,z}/(\pi_1^{e_1}\pi_2^{e_2})$ are evidently just the (π_i), which shows X_0 has no embedded point at z.

Since D_0 is a disjoint union of regular closed points, by (∗∗) and the work that immediately precedes it we have $\mathrm{H}^0(C) \simeq \mathrm{H}^0(U_0)$ and an exact sequence

$$0 \longrightarrow \mathrm{H}^1(C) \longrightarrow \mathrm{H}^1(U_0) \xrightarrow{\partial_{D_0}} \mathrm{H}^0(\kappa(D_0), -1) \longrightarrow \mathrm{H}^2(C) \longrightarrow \cdots$$

Thus we have a commutative ladder

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(X) \arrow[r] \arrow[d] & \mathrm{H}^1(U) \arrow[r, "\partial_D"] \arrow[d] & \mathrm{H}^0(D,-1) \arrow[r] \arrow[d] & \mathrm{H}^2(X) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & \mathrm{H}^1(C) \arrow[r] & \mathrm{H}^1(U_0) \arrow[r, "\partial_{D_0}"] & \mathrm{H}^0(D_0,-1) \arrow[r] & \mathrm{H}^2(C) \arrow[r] & \cdots
\end{tikzcd}$$

Since R is complete, $\mathrm{H}^q(X) \to \mathrm{H}^q(C)$ and $\mathrm{H}^q(D,-1) \to \mathrm{H}^q(D_0,-1)$ are isomorphisms for $q \geq 0$ by proper base change ([28, VI.Corollary 2.7]). Therefore, in light of the isomorphisms in degree zero and the 5-lemma in degree $q \geq 1$, we obtain isomorphisms

$$\mathrm{H}^q(U) \xrightarrow{\sim} \mathrm{H}^q(U_0)$$

for $q \geq 0$. Let $\widetilde{U}$ denote the inverse limit over all these open sets U (this is a scheme by [20, 8.2.3]). Then $\mathrm{H}^q(\widetilde{U})$ is the direct limit of the $\mathrm{H}^q(U)$ by [28, III.1.16], and since the direct limit functor is exact we have an isomorphism $\mathrm{H}^q(\widetilde{U}) \xrightarrow{\sim} \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}})$. Composing the inverse with $\mathrm{H}^q(\widetilde{U}) \to \mathrm{H}^q(K(X))$ yields our lift

$$\lambda : \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}}) \longrightarrow \mathrm{H}^q(K(X))$$

The commutative diagram (4.7) follows by applying cohomology to the diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathrm{O}_{C,\mathcal{S}} \arrow[r] & \widetilde{U} & \operatorname{Spec} K(X) \arrow[l] \\
\operatorname{Spec} \kappa(C_i) \arrow[r] \arrow[u] & \operatorname{Spec} \mathrm{O}_{K(X)_{C_i}} \arrow[u] & \operatorname{Spec} K(X)_{C_i} \arrow[l] \arrow[u]
\end{tikzcd}$$

incorporating the isomorphisms induced by the upper and lower left horizontal arrows. If $\alpha_0 \in \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}})$ and $\alpha = \lambda(\alpha)$, then since $\widetilde{U}$ contains $\mathcal{S}$ and the generic points of the C_i, α is defined at these points, and the formula $\alpha(C_i) = \mathrm{res}_{\mathrm{O}_{C,\mathcal{S}}|\kappa(C_i)}(\alpha_0)$ follows immediately from (4.7), proving (a).

If D is a horizontal prime divisor not in $\mathscr{D}_{\mathcal{S}}$, then the generic point $\operatorname{Spec}\kappa(D)$ is contained in $\widetilde{U}$, hence the map $\mathrm{H}^q(\widetilde{U}) \to \mathrm{H}^q(K(X)_D)$ factors through $\mathrm{H}^q(\mathrm{O}_{K(X)_D})$, which shows $\partial_D \cdot \lambda = 0$. Thus the ramification locus of any element in the image of λ must be contained in $\mathscr{D}_{\mathcal{S}}$, proving (b). Now if $D \in \mathscr{D}_{\mathcal{S}}$ is prime and $z = D \cap C$ then D is the prime spectrum of a complete local ring with residue field $\kappa(z)$, and the isomorphism

$$\mathrm{H}^{q-1}(D,-1) \xrightarrow{\sim} \mathrm{H}^{q-1}(z,-1) = \mathrm{H}^{q-1}(\kappa(z),-1)$$

is the standard identification. Thus the formula $\partial_D \cdot \lambda = \inf_{\kappa(z)|\kappa(D)} \cdot \partial_z$ is immediate by the commutative ladder of Gysin sequences above, proving (c).

Suppose $\alpha = \lambda(\alpha_0)$ has ramification locus D_α, then $D_\alpha \in \mathscr{D}_{\mathcal{S}}$. Set $U = X - D_\alpha$. If α_0 is unramified at a point z, then α is unramified at every prime divisor D lying over z. For if $D \in \mathscr{D}_{\mathcal{S}}$ then $\partial_D(\alpha) = \inf(\partial_z(\alpha_0))$ by the formula just proved, and if $D \notin \mathscr{D}_{\mathcal{S}}$ then $\partial_D(\alpha) = 0$ since $D_\alpha \in \mathscr{D}_{\mathcal{S}}$. Thus if α_0 is unramified at z, and D is a prime divisor lying over z, then U contains D. The maps $z = \operatorname{Spec}\kappa(z) \to U_0$ and $D \to U$ then induce a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^q(U) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & \mathrm{H}^q(D) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{res}"] & \mathrm{H}^q(\kappa(D)) \\
\mathrm{H}^q(U_0) \arrow[r, "\mathrm{res}"] & \mathrm{H}^q(z) \arrow[ur, "\inf"'] &
\end{tikzcd}$$

Both vertical down-arrows are isomorphisms by proper base change. The inverse of the left one is λ by definition, and the composition of the inverse of the right one and the restriction $\mathrm{H}^q(D) \to \mathrm{H}^q(\kappa(D))$ is inflation, as shown. Since $\kappa(D)$ is complete, the top composition of horizontal restrictions factors through the restriction $\mathrm{H}^q(U) \to \mathrm{H}^q(\widehat{\mathrm{O}}_{X,D})$ and the bottom factors through the restriction $\mathrm{H}^q(U_0) \to \mathrm{H}^q(\widehat{\mathrm{O}}_{C,z})$. Since these are restriction maps, the images of α and α_0 are the values $\alpha(D)$ and $\alpha_0(z)$. We conclude $\inf_{\kappa(z)|\kappa(D)}(\alpha_0(z)) = \alpha(D)$, as in (d).

■

4.8. By weak approximation [34, Lemma] there exists a $\pi \in K(X)$ such that

$$\operatorname{div}\pi = C + E$$

where E contains no components of C, and avoids any finite set of points $\mathcal{N}$. We fix such a π for $\mathcal{N}$ containing $\mathcal{S}$. For each i, the choice of π determines a noncanonical "Witt" isomorphism

$$\mathrm{H}^q(\kappa(C_i)) \oplus \mathrm{H}^{q-1}(\kappa(C_i),-1) \xrightarrow{\sim} \mathrm{H}^q(K(X)_{C_i})$$

Taking (α,θ) to $\alpha + (\pi)\cdot\theta$, where α and θ are inflated from $\kappa(C_i)$ to $K(X)_{C_i}$, (π) is the image of π in $\mathrm{H}^1(K(X)_{C_i},\mu_n)$, and $(\pi)\cdot\theta$ is the cup product. Although we cannot in general lift all of $\bigoplus_i \mathrm{H}^q(K(X)_{C_i})$ to $\mathrm{H}^q(K(X)$, we can now prove the following.

Corollary 4.9. *Let* (π) *denote the image of* π *in* $\mathrm{H}^1(K(X),\mu_n)$. *The choice of* $\mathscr{D}_{\mathcal{S}}$ *and* π *determines a homomorphism for* $q\geq 1$,

$$\lambda : \mathrm{H}^q(\mathrm{O}_{C,\mathcal{S}},\Lambda)\oplus \mathrm{H}^{q-1}(\mathrm{O}_{C,\mathcal{S}},\Lambda(-1)) \longrightarrow \mathrm{H}^q(K(X),\Lambda)$$
$$(\alpha_0,\theta_0)\longmapsto \lambda(\alpha_0)+(\pi)\cdot\lambda(\theta_0)$$

such that $(\bigoplus_i \mathrm{res}_{K(X)|K(X)_{C_i}})\cdot\lambda = \bigoplus_i(\inf{}_{\kappa(C_i)|K(X)_{C_i}}\cdot \mathrm{res}_{\mathrm{O}_{C,\mathcal{S}}|\kappa(C_i)})$.

Proof. This is an immediate consequence of Theorem 4.6. ■

Remark 4.10. a) Theorem 4.6 and Corollary 4.9 apply with obvious amendments to the case where X is not connected. For if $X=\coprod_k X_k$ is a decomposition into connected components, then $K(X)=\prod_k K(X_k)$, $X_0=\coprod_k (X_k)_0$, $\mathrm{O}_{X_0,\mathcal{S}}=\prod_k \mathrm{O}_{(X_k)_0,\mathcal{S}_k}$ (where $\mathcal{S}_k=\mathcal{S}\cap X_k$), all of the cohomology groups break up into direct sums, and we define the map λ to be the direct sum of the maps on the summands. This will come up in the next section.

b) If X is smooth, then $\mathcal{S}$ is empty, and $\mathrm{O}_{C,\mathcal{S}}=\kappa(C)$. By Witt's theorem we have $\mathrm{H}^q(K(X)_C,\Lambda)\simeq \mathrm{H}^q(\kappa(C),\Lambda)\oplus\mathrm{H}^{q-1}(\kappa(C),\Lambda(-1))$, and we obtain a map

$$\lambda : \mathrm{H}^q(K(X)_C,\Lambda)\longrightarrow \mathrm{H}^q(K(X),\Lambda)$$

that splits the restriction map. This is the map of [10].

4.11. Completely split characters. In [30, 2.1] Saito defines a completely split covering of a noetherian scheme X to be a finite étale cover $Y\to X$ such that $Y\times_X \operatorname{Spec}\kappa(x)=\coprod \operatorname{Spec}\kappa(x)$, for all closed points $x\in X$. We abuse Saito's terminology (see Remark(4.13) below) and in the setup of (2.3) denote by $\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ the kernel of the map $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}},\mathbb{Z}/n)\to\mathrm{H}^1(\kappa(C),\mathbb{Z}/n)$ in Lemma 4.4. If $\beta\in\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ then $\partial_z(\beta)=0$ for all closed points $z\in C-\mathcal{S}$ since ∂_z factors through $\kappa(C_i)_z$. Therefore β is defined on C, hence $\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)\leq\mathrm{H}^1(C,\mathbb{Z}/n)$. Let $\mathrm{H}^1_{\mathrm{cs}}(X,\mathbb{Z}/n)$ denote the preimage of $\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ under the proper base change isomorphism.

Proposition 4.12. *Assume the setup of (2.3). Then elements of* $\mathrm{H}^1_{\mathrm{cs}}(C,\mathbb{Z}/n)$ *are trivial at all points of* C, *and the nontrivial elements of* $\mathrm{H}^1_{\mathrm{cs}}(X,\mathbb{Z}/n)$ *are trivial at all points of* X *except for the generic point* $\operatorname{Spec}K(X)$, *where they are nontrivial.*

Proof. Suppose $\beta_0\in\mathrm{H}^1_{\mathrm{cs}}(C)$. Then β_0 is trivial at each generic point of C by definition of $\mathrm{H}^1_{\mathrm{cs}}(C)$. If $z\in C$ is a closed point lying on the irreducible component C_i then the map $\mathrm{H}^1(C)\to\mathrm{H}^1(\kappa(z))$ factors through $\mathrm{H}^1(C_i)$. Since C_i is regular the map $\mathrm{H}^1(C_i)\to\mathrm{H}^1(\kappa(C_i))$ is injective by purity, and consequently $\beta_0(z)=0$ by definition. Thus the elements of $\mathrm{H}^1_{\mathrm{cs}}(C)$ are trivial at all points of C.

Suppose $\beta=\lambda(\beta_0)\in\mathrm{H}^1_{\mathrm{cs}}(X)$. If $x\in X$ is a generic point of some irreducible component C_i of C then the image of β in $\mathrm{H}^1(\kappa(C_i))$ is zero since the map $\mathrm{H}^1_{\mathrm{cs}}(X,\mathbb{Z}/n)\to\mathrm{H}^1(\kappa(C_i))$ factors through $\mathrm{H}^1_{\mathrm{cs}}(C)$. If x is the generic point of a horizontal divisor D with closed point z then $\beta(D)=\inf{}_{\kappa(z)|\kappa(D)}(\beta_0(z))$ by Theorem 4.6(d), and this is zero since $\beta_0(z)=0$. If z is a closed point of X then z is on C, and the map $\mathrm{H}^1_{\mathrm{cs}}(X)\to\mathrm{H}^1(\kappa(z))$ factors through $\mathrm{H}^1_{\mathrm{cs}}(C)$, hence β is trivial at z. Finally, since X is regular the map $\mathrm{H}^1(X)\to\mathrm{H}^1(K(X))$ is injective by purity, hence β is nontrivial at the generic point of X. □

Remark 4.13. Proposition 4.12 shows the elements of $\mathrm{H}^1_{\mathrm{cs}}(C, \mathbb{Z}/n)$ are completely split in the sense of [30]. However, in the general case our $\mathrm{H}^1_{\mathrm{cs}}(C, \mathbb{Z}/n)$ does not account for elements that are split at every closed point but nontrivial at generic points of C. This is not an issue if k is finite as shown by Saito in [30, Theorem 2.4], since then the C_i have no nontrivial completely split covers, essentially by Cebotarev's density theorem (see [29, Lemma 1.7]).

5. Index Calculation in the Brauer Group

5.1. Cyclic Covers. If U is any scheme, and $\bar{u}$ is a geometric point, the fiber functor defines a category equivalence between (finite) étale covers of U and finite continuous $\pi_1(U, \bar{u})$-sets, yielding a canonical isomorphism

$$\mathrm{H}^1(U, \mathbb{Z}/n) \simeq \mathrm{H}^1(\pi_1(U, \bar{u}), \mathbb{Z}/n) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1(U, \bar{u}), \mathbb{Z}/n)$$

(see [13, I.2.11]). If $\theta \in \mathrm{H}^1(U, \mathbb{Z}/n)$, we will write $U[\theta]$ for the finite cyclic étale cover determined by θ. If $U = \operatorname{Spec} A$ is affine, we will write $A[\theta]$ for the corresponding ring, or $A(\theta)$ if A is a field. If U is a connected normal scheme, and $\theta \in \mathrm{H}^1(U, \mathbb{Z}/n)$ has order m, then $U[\theta]$ is a disjoint sum of n/m connected $\mathbb{Z}/m$-Galois covers of U.

Lemma 5.2. *Assume the setup of (2.3). Let $\theta_0 \in \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, \mathbb{Z}/n)$ be a (tamely ramified) character with ramification divisor D_0 on C. Then the (tame) ramification divisor of $\theta = \lambda(\theta_0)$ is the distinguished lift $D \in \mathscr{D}_{\mathcal{S}}$ of D_0 on X, and θ defines a tamely ramified cover $\rho : Y \to (X, D)$ as in Lemma 3.2. Restriction to C yields a tamely ramified cover $\rho_0 : C_Y \to (C, D_0)$ such that $\mathrm{O}_{C_Y,\mathcal{S}_Y} = \mathrm{O}_{C,\mathcal{S}}[\theta_0]$, and the reduced closed fiber C_Y of Y is the normalization of C in $\kappa(C)(\theta_0)$.*

Proof. The lift θ is tamely ramified with respect to D by Theorem 4.6. Let Y be the normalization of X in $L = K(X)(\theta)$. Since D is in $\mathscr{D}_{\mathcal{S}}$ and X/R satisfies the setup of (2.3), by Lemma 3.2 $\rho : Y \to (X, D)$ is a tamely ramified cover, Y/R is a regular relative curve with reduced closed fiber C_Y, the irreducible components of C_Y are regular with singular points $\mathcal{S}_Y$, and D_Y is in $\mathscr{D}_{Y,\mathcal{S}_Y}$.

Let $U = X - D$, $V = U \times_X Y$, $U_0 = U \times_X X_0$ and $V_0 = V \times_Y Y_0$. Then $\theta_0 \in \mathrm{H}^1(U_0)$. We have $\mathrm{H}^1(U) \simeq \mathrm{Hom}(\pi_1(U), \mathbb{Z}/n)$ and $\mathrm{H}^1(U_0) \simeq \mathrm{Hom}(\pi_1(U_0), \mathbb{Z}/n)$, and the restriction map $\mathrm{res} : \mathrm{H}^1(U) \to \mathrm{H}^1(U_0)$, which sends θ to θ_0, is induced by the natural map $\pi_1(U_0) \to \pi_1(U)$, which is induced on covers by $W \mapsto W \times_U U_0$. Therefore $V_0 = U_0[\theta_0]$.

We show $\mathrm{O}_{C_Y,\mathcal{S}_Y} = \mathrm{O}_{C,\mathcal{S}}[\theta_0]$. If $U_0 = \operatorname{Spec} A_0 \subset C - D_0$ is a dense affine open subset of C containing $\mathcal{S}$, then its preimage in C_Y is a dense affine open subset $V_0 = \operatorname{Spec} B_0$ containing $\mathcal{S}_Y$. By base change we have $B_0 = A_0[\theta_0]$, and $S^{-1}B_0 = \mathrm{O}_{C,\mathcal{S}}[\theta_0]$, where $S = A_0 - \left(\bigcup_{x\in\mathcal{S}} \mathfrak{m}_x\right)$ is the multiplicative set defining $\mathrm{O}_{C,\mathcal{S}}$. Since $\mathcal{S}_Y = \rho^{-1}\mathcal{S}$, the saturation T of S in B_0 is $T = B_0 - \left(\bigcup_{y\in\mathcal{S}_Y} \mathfrak{m}_y\right)$, which shows $S^{-1}B_0 = \mathrm{O}_{C_Y,\mathcal{S}_Y}$. Therefore $\mathrm{O}_{C_Y,\mathcal{S}_Y} = \mathrm{O}_{C,\mathcal{S}}[\theta_0]$, and it follows immediately that $\mathrm{O}_{C_Y,\mathcal{S}_Y} = \mathrm{O}_{C,\mathcal{S}}[\theta_0]$.

The map $C_Y \to C$ is finite by base change, and since each irreducible component of C_Y is regular by Lemma 3.1 each irreducible component of C_Y is the normalization of a component of C in a field extension which is a direct factor of $\kappa(C_Y)$. Equivalently, C_Y is the normalization of C in $\kappa(C)(\theta_0)$, by [18, 6.3.7]. This completes the proof.

■

5.3. Index. Assume the setup of (2.3) with $R = \mathbb{Z}_p$ and $\Lambda = \mu_n$. Fix $\alpha_C \in \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}})$ and $\theta_C \in \Gamma \leq \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, -1)$, and write

$$\alpha_C = (\alpha_{C_1}, \ldots, \alpha_{C_m}) \in \mathrm{H}^2(\kappa(C))$$
$$\theta_C = (\theta_{C_1}, \ldots, \theta_{C_m}) \in \mathrm{H}^1(\kappa(C), -1)$$

as per Lemma 4.4(b). Suppressing the inflation maps, we form the element

$$\gamma_C = \alpha_C + (\pi) \cdot \theta_C = (\alpha_{C_1} + (\pi) \cdot \theta_{C_1}, \ldots, \alpha_{C_m} + (\pi) \cdot \theta_{C_m}) \in \bigoplus_{i=1}^{m} \mathrm{H}^2(K(X)_{C_i})$$

where π is as in (4.8). Let $\theta = \lambda(\theta_C)$, and let Y be the normalization of X in $K(X)(\theta)$, as in Lemma 5.2. Then C_Y is the reduced closed fiber of Y, and we write $C_{i,Y}$ for the preimage of C_i, so that $\kappa(C_{i,Y}) = \kappa(C_i)(\theta_{C_i})$. Thus

$$\alpha_{C_Y} = (\alpha_{C_{1,Y}}, \ldots, \alpha_{C_{m,Y}}) \in \bigoplus_{i=1}^{m} \mathrm{H}^2(\kappa(C_{i,Y}))$$

Note $\kappa(C_{i,Y})$ is a product of the function fields of the irreducible components of $C_{i,Y}$. By the (well-known) Nakayama-Witt index formula,

$$\mathrm{ind}(\alpha_{C_i} + (\pi) \cdot \theta_{C_i}) = |\theta_{C_i}| \cdot \mathrm{ind}(\alpha_{C_{i,Y}})$$

We have $|\theta_C| = \mathrm{lcm}_i\{|\theta_{C_i}|\}$ by Lemma 4.4(b). We now *define*

$$\mathrm{ind}(\alpha_{C_Y}) \overset{\mathrm{df}}{=} \mathrm{lcm}_i\{\mathrm{ind}(\alpha_{C_{i,Y}})\}$$
$$\mathrm{ind}(\gamma_C) \overset{\mathrm{df}}{=} |\theta_C| \cdot \mathrm{ind}(\alpha_{C_Y})$$

Theorem 5.4. *Assume the setup of (2.3) with $R = \mathbb{Z}_p$. Let $\Gamma \leq \mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}})$ be as in Lemma 4.4(b). Then the map $\lambda : \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \oplus \Gamma \to \mathrm{H}^2(K(X))$ preserves index.*

Proof. We may assume X is connected. We identify Γ with the image of $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, -1)$ in $\mathrm{H}^1(\kappa(C), -1)$, as in Lemma 4.4(b), and adopt the notation of (5.3). Set $\gamma = \lambda(\gamma_C)$, $\alpha = \gamma(\alpha_C)$, and $\theta = \lambda(\theta_C)$, so that $\gamma = \alpha + (\pi) \cdot \theta$ as in Corollary 4.9. By restricting to connected components if necessary we may assume that Y is connected, hence that $K(Y)$ is a field. Even so, the cyclic-Galois étale $\kappa(C_i)$-algebra $\kappa(C_{i,Y})$ may not be a field. Since θ_C is in $\mathrm{H}^1(\mathrm{O}_{C,\mathcal{S}}, -1)$ the ramification divisor D_0 of θ_C avoids $\mathcal{S}$, and the distinguished lift $D \in \mathscr{D}_{\mathcal{S}}$ of D_0 is the ramification divisor of θ by Theorem 4.6. By Lemma 5.2 $\rho : Y \to (X, D)$ is a cyclic tamely ramified cover, and by Lemma 3.1 Y/R satisfies the properies of (2.3), with reduced closed fiber C_Y, $\mathcal{S}_Y = \rho^{-1}\mathcal{S}$ the singular points of C_Y, and $\mathscr{D}_Y$ generated by $D_{Y,\mathrm{red}}$ and the preimages of the other distinguished divisors of X.

The index of γ cannot exceed $|\theta|\mathrm{ind}(\alpha_Y)$. For if $M/K(Y)$ is a separable maximal subfield of the division algebra associated with α_Y, then $M/K(X)$ splits γ, and has degree $|\theta|\mathrm{ind}(\alpha_Y)$. Since in our case $|\theta| = [K(Y) : K(X)] = [\kappa(C_Y) : \kappa(C)] = |\theta_C|$, to prove the theorem it is enough to prove $\mathrm{ind}(\alpha_Y) = \mathrm{ind}(\alpha_{C_Y})$.

By Lemma 5.2 $\mathrm{O}_{C,\mathcal{S}}[\theta_C] = \mathrm{O}_{C_Y,\mathcal{S}_Y}$. Each $\kappa(C_{i,Y}) = \kappa(C_i)(\theta_{C_i})$ is a product of global fields, and by class field theory the division algebra associated with the restriction of $\alpha_{C_{i,Y}}$ to each field component is cyclic. Since α_{C_Y} is in $\mathrm{H}^2(\mathrm{O}_{C_Y,\mathcal{S}_Y})$ (by Lemma 5.2 and Lemma 4.2), $\alpha_{C_{i,Y}}$ is unramified at $\mathcal{S}_Y$. By Grunwald-Wang's

theorem there exists a tuple $\psi_{C_Y} = (\psi_{C_{1,Y}}, \ldots, \psi_{C_m, Y}) \in \bigoplus_i \mathrm{H}^1(\kappa(C_{i,Y}), -1)$ such that $|\psi_{C_{i,Y}}| = \operatorname{ind}(\alpha_{C_{i,Y}})$, $\kappa(\psi_{C_{i,Y}})$ splits $\alpha_{C_{i,Y}}$, and such that the $\psi_{C_{i,Y}}$ are unramified and equal at the local fields defined by the singular points $\mathcal{S}_Y$. Then ψ_{C_Y} comes from $\mathrm{H}^1(\mathrm{O}_{C_Y, \mathcal{S}_Y}, -1)$ by Lemma 4.2, and $|\psi_{C_Y}| = \operatorname{ind}(\alpha_{C_Y})$.

By Theorem 4.6 (and Remark 4.10(a) if Y is not connected) we have a map $\lambda_Y : \mathrm{H}^1(\mathrm{O}_{C_Y, \mathcal{S}_Y}, -1) \to \mathrm{H}^1(K(Y), -1)$. Since the distinguished divisors on Y are the (reduced) preimages of those on X, λ_Y is compatible with λ and the residue maps. Set $\psi = \lambda_Y(\psi_{C_Y})$, and let D_ψ denote the distinguished lift of the ramification divisor $D_{\psi_{C_Y}}$ of ψ_{C_Y} on C_Y. By Lemma 5.2, ψ determines a cyclic tamely ramified cover $\sigma : Z \to (Y, D_\psi)$ with reduced closed fiber C_Z, such that C_Z is cyclic and tamely ramified over $(C_Y, D_{\psi_{C_Y}})$, inducing ψ_{C_Y}. Since $\kappa(C_{i,Z})$ splits $\alpha_{C_{i,Y}}$, $\alpha_{C_Z} = 0$.

Again we may assume Z is connected. By construction, $[K(Z) : K(Y)] = |\psi| = |\psi_{C_Y}| = [\kappa(C_Z) : \kappa(C_Y)] = \operatorname{ind}(\alpha_{C_Y})$. By (4.7) we have $\operatorname{ind}(\alpha_{C_Y}) \leq \operatorname{ind}(\alpha_Y)$, and it remains to show $\alpha_Z = 0$. It is then enough to show α_Z is unramified with respect to all Weil divisors on Z, by [10, Lemma 3.5].

Let $D' = D \cup \rho(D_\psi)$. Since $D_\psi \in \mathscr{D}_{Y, \mathcal{S}_Y}$, $\rho(D_\psi) \in \mathscr{D}_{\mathcal{S}}$, and the composition $\rho' : Z \to (X, D')$ is a tamely ramified cover. Since X is regular, ρ' is (finite and) flat by Lemma 3.1, and so the image of any prime divisor J of Z is a prime divisor $\rho'(J) = I$ of X. By the functoriality of the residue maps, α_Z can only be ramified at prime divisors lying over irreducible components of D_α. By Theorem 4.6(a) D_α is in $\mathscr{D}_{\mathcal{S}}$, and the divisors of Z lying over D_α are distinguished by Lemma 3.1(c). Thus it is enough to show that α_Z is unramified at these distinguished divisors. Clearly we may assume D_α is irreducible.

Let $E \subset Z$ be a (distinguished) prime divisor lying over $D_\alpha \in \mathscr{D}_{\mathcal{S}}$. Since $D_\alpha \cap D' = \varnothing$ or $D_\alpha \subset D'$, we have $e(v_E/v_{D_\alpha}) = e(v_{E_0}/v_{D_{\alpha_C}}) = e$ for some $e \geq 1$, by Lemma 3.3. By Lemma 4.6 and the functorial behavior of the residue and restriction maps, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \arrow[r, "\lambda"] \arrow[d, "\partial_{D_{\alpha_C}}"'] \arrow[dr, phantom, "1"] & \mathrm{H}^2(K(X)) \arrow[r, "\mathrm{res}"] \arrow[d, "\partial_{D_\alpha}"'] \arrow[dr, phantom, "2"] & \mathrm{H}^2(K(Z)) \arrow[d, "\partial_E"] \\
\mathrm{H}^1(\kappa(D_{\alpha_C}), -1) \arrow[r] & \mathrm{H}^1(\kappa(D_\alpha), -1) \arrow[r, "e\cdot\mathrm{res}"] & \mathrm{H}^1(\kappa(E), -1) \\
& \mathrm{H}^1(\kappa(D_{\alpha_C}), -1) \arrow[r, "e\cdot\mathrm{res}"] \arrow[u, "\mathrm{inf}"] \arrow[ur, phantom, "3"] & \mathrm{H}^1(\kappa(E_0), -1) \arrow[u, "\mathrm{inf}"'] \\
& \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \arrow[r, "\mathrm{res}"] \arrow[u, "\partial_{D_{\alpha_C}}"] \arrow[ur, phantom, "4"] & \mathrm{H}^2(\mathrm{O}_{C_Z, \mathcal{S}_Z}) \arrow[u, "\partial_{E_0}"']
\end{tikzcd}$$

Since $\alpha = \lambda(\alpha_C)$, $\partial_E(\alpha_Z) = e\cdot(\partial_{D_{\alpha_C}}(\alpha_C))_{\kappa(E)}$ by squares (1) and (2), and by square (4), $\partial_{E_0}(\alpha_{C_Z}) = e \cdot (\partial_{D_{\alpha_C}}(\alpha_C))_{\kappa(E_0)}$. Therefore $\partial_E(\alpha_Z) = \inf_{\kappa(E_0)|\kappa(E)}(\partial_{E_0}(\alpha_{C_Z})$ by square (3). Since $\alpha_{C_Z} = 0$, we conclude $\partial_E(\alpha_Z) = 0$, as desired. This proves the theorem.

■

6. Noncrossed Products and Indecomposable Division Algebras

A (finite-dimensional) division algebra D central over a field F is called a *noncrossed product* if it has no Galois maximal subfield. Its algebra structure then cannot be given by a Galois 2-cocycle, counter to almost all known division algebra constructions (see [24] for a construction of a noncrossed product). Noncrossed product division algebras were long conjectured to be fictional, until they were shown to exist by Amitsur in [1].

We say D is *indecomposable* if it does not contain a subalgebra that is also central over F, or equivalently if it is not an F-tensor product of two nontrivial F-division algebras. It is not hard to show that all division algebras of composite period are decomposable, and that all division algebras of equal prime-power period and index are indecomposable, but it is nontrivial to construct indecomposable division algebras of unequal prime-power period and index. The first examples appeared in [3] and in [31]. For additional discussion of either of these topics, see almost any survey treating open problems on division algebras, such as [5], [2], or [32].

We can use Theorem 5.4 to prove the existence of noncrossed product and indecomposable division algebras over the function field F of any p-adic curve $X_{\mathbb{Q}_p}$. Noncrossed products over $K(t)$ for K a local field were first constructed in [8], and then constructed more sytematically over the function field of a smooth relative $\mathbb{Z}_p$-curve in [10]. Indecomposable division algebras of unequal period and index were also constructed in [10], over the same types of fields. Modulo gluing, the method we use below is the same as the one used in [10, Theorem 4.3, Corollary 4.8].

Theorem 6.1. *Let $F/\mathbb{Q}_p$ be a finitely generated field extension of transcendence degree one. Let $X/\mathbb{Z}_p$ be a regular relative curve with function field F, let C_i be a reduced irreducible component of the closed fiber, let $\ell \neq p$ be a prime, and let r and s be numbers that are maximal such that $\mu_{\ell^r} \subset \kappa(C_1)$ and $\mu_{\ell^s} \subset \kappa(C_1)(\mu_{\ell^{r+1}})$. Then there exist noncrossed product F-division algebras of period and index as low as ℓ^{s+1} if $r = 0$, and ℓ^{2r+1} if $r \neq 0$.*

Proof. We may assume (without changing r and s) that C has regular irreducible components, at most two of which meet at any closed point of X. The idea is to use the (known) existence of such algebras over the fields F_{C_i}, modify the construction to produce a class in $\mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \oplus \Gamma$, and then apply Theorem 4.6 and Theorem 5.4 to prove existence over F.

By [10, Theorem 4.7], if F admits a smooth X then there exist noncrossed product division algebras over F_C of period and index as low as ℓ^{s+1} if $r = 0$, and ℓ^{2r+1} if $r > 0$. The resulting Brauer class has the form $\alpha_C + (\pi) \cdot \theta_C \in \mathrm{H}^2(F_C)$, where $\alpha_C \in \mathrm{H}^2(\kappa(C))$ and $\theta_C \in \mathrm{H}^1(\kappa(C), -1)$. A look at the construction, which proceeds exactly as in [6, Theorem 1], shows we may pre-assign values at any finite set of points of C. Thus we may thus produce a noncrossed product F_{C_1}-division algebra with class $\gamma_{C_1} = \alpha_{C_1} + (\pi) \cdot \theta_{C_1}$ of the desired period and index, and elements $\gamma_{C_i} = \alpha_{C_i} + (\pi) \cdot \theta_{C_i}$ for $i > 1$ that glue as in Lemma 4.2. Let

$$\gamma_C = (\alpha_{C_1} + (\pi) \cdot \theta_{C_1}, \ldots, \alpha_{C_m} + (\pi) \cdot \theta_{C_m}) \in \mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \oplus (\pi) \cdot \Gamma \leq \bigoplus_i \mathrm{H}^2(F_{C_i})$$

Then γ_C lifts to $\gamma = \lambda(\gamma_C) \in \mathrm{H}^2(F)$ by Theorem 4.6. By Theorem 5.4, $\mathrm{ind}(\gamma) = \mathrm{ind}(\gamma_C) = \ell^{s+1}$ if $r = 0$, and ℓ^{2r+1} if $r \neq 0$. It is clear that the F-division algebra

D associated to γ is a noncrossed product, since any Galois maximal subfield of D over F could be constructed over F_{C_1}, contradicting the fact that $D \otimes_F F_{C_1}$ is a noncrossed product F_{C_1}-division algebra.

■

Theorem 6.2. *Let $F/\mathbb{Q}_p$ be a finitely generated field extension of transcendence degree one, and let $\ell \neq p$ be a prime. Then there exist indecomposable F-division algebras of (period,index)= (ℓ^a, ℓ^b), for any numbers a and b satisfying $1 \leq a \leq b \leq 2a-1$.*

Proof. Let X, C, C_i, and $\mathcal{S}$ be as in Theorem 6.1. The construction over F_{C_i} is exactly as in [10, Proposition 4.2] and [7], and we merely have to observe that we may assume all of the data in the constructed class $\gamma_{C_i} = \alpha_{C_i} + (\pi) \cdot \theta_{C_i} \in \mathrm{H}^2(F_{C_i})$ is trivial at the singular points $\mathcal{S} \cap C_i$, so that by Lemma 4.2, we may construct a class $\gamma_C = \alpha_C + (\pi) \cdot \theta_C$ in $\mathrm{H}^2(\mathrm{O}_{C,\mathcal{S}}) \oplus (\pi) \cdot \Gamma \leq \bigoplus_i \mathrm{H}^2(F_{C_i})$ whose i-th component is $\alpha_{C_i} + (\pi) \cdot \theta_{C_i}$. This class lifts to a class $\gamma = \lambda(\gamma_C)$ by Theorem 4.6, and $\mathrm{ind}(\gamma) = \mathrm{ind}(\gamma_C)$ by Theorem 5.4. Since the indexes are the same, the division algebra D associated to γ is indecomposable, since any decomposition would extend to $D_{C_i} = D \otimes_F F_{C_i}$, contradicting the construction of γ_{C_i}.

■

References

[1] S. Amitsur. On central division algebras. *Israel J. Math.*, 12:408–422, 1972.

[2] S. Amitsur. Division algebras. A survey. In *Algebraists' homage: papers in ring theory and related topics (New Haven, Conn., 1981)*, volume 13 of *Contemp. Math.*, pages 3–26. Amer. Math. Soc., Providence, R.I., 1982.

[3] S. Amitsur, L. Rowen, and J.-P. Tignol. Division algebras of degree 4 and 8 with involution. *Israel J. Math.*, 33:133–148, 1979.

[4] M. F. Atiyah and I. G. Macdonald. *Introduction to Commutative Algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[5] A. Auel, E. Brussel, S. Garibaldi, and U. Vishne. Open problems on central simple algebras. *Transform. Groups*, 16(1):219–264, March 2011.

[6] E. Brussel. Noncrossed products and nonabelian crossed products over $\mathbf{Q}(t)$ and $\mathbf{Q}((t))$. *Amer. J. Math.*, 117(2):377–393, 1995.

[7] E. Brussel. Decomposability and embeddability of discretely Henselian division algebras. *Israel J. Math.*, 96(part A):141–183, 1996.

[8] E. Brussel. Noncrossed products over $k_{\mathfrak{p}}(t)$. *Trans. Amer. Math. Soc.*, 353(5):2115–2129 (electronic), 2001.

[9] E. Brussel. On Saltman's p-adic curves papers. In *Quadratic forms, linear algebraic groups, and cohomology*, volume 18 of *Dev. Math.*, pages 13–39. Springer, New York, 2010.

[10] E. Brussel, K. McKinnie, and E. Tengan. Indecomposable and noncrossed product division algebras over function fields of smooth p-adic curves. *Adv. in Math.*, 226:4316–4337, 2011.

[11] E. Brussel and E. Tengan. Division algebras of prime period $\ell \neq p$ over function fields of p-adic curves. arxiv.org:1003.1906, 2010.

[12] J.-L. Colliot-Thélène. Birational invariants, purity and the Gersten conjecture. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, volume 58 of *Proc. Sympos. Pure Math.*, pages 1–64. Amer. Math. Soc., Providence, RI, 1995.

[13] E. Freitag and R. Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné.

[14] L. Fuchs. *Infinite Abelian Groups. Vol. I.* Pure and Applied Mathematics, Vol. 36. Academic Press, New York, 1970.

[15] K. Fujiwara. A proof of the absolute purity conjecture (after Gabber). In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 153–183. Math. Soc. Japan, Tokyo, 2002.

[16] S. Garibaldi, A. Merkurjev, and J.-P. Serre. *Cohomological invariants in Galois cohomology*, volume 28 of *University Lecture Series*. Amer. Math. Soc., 2003.

[17] A. Grothendieck. Éléments de géométrie algébrique. I. Le langage des schémas. *Inst. Hautes Études Sci. Publ. Math.*, (4):228, 1960.

[18] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[19] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II. *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[20] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. *Inst. Hautes Études Sci. Publ. Math.*, (28):255, 1966.

[21] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[22] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)].

[23] A. Grothendieck and J. P. Murre. *The tame fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme*. Lecture Notes in Mathematics, Vol. 208. Springer-Verlag, Berlin, 1971.

[24] T. Hanke. A twisted Laurent series ring that is a noncrossed product. *Israel J. Math.*, 150:199–203, 2005.

[25] S. Lang. *Algebra*. Springer-Verlag, third edition, 2002.

[26] Q. Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[27] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[28] J. Milne. *Étale Cohomology*. Princeton University Press, 1980.

[29] W. Raskind. Abelian class field theory of arithmetic schemes. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, volume 58 of *Proc. Sympos. Pure Math.*, pages 85–187. Amer. Math. Soc., Providence, RI, 1995.

[30] S. Saito. Class field theory for curves over local fields. *J. Number Theory*, 21(1):44–80, 1985.

[31] D. Saltman. Indecomposable division algebras. *Comm. Algebra*, 7(8):791–817, 1979.

[32] D. Saltman. Finite dimensional division algebras. In *Azumaya algebras, actions, and modules*, volume 124 of *Contemp. Math.*, pages 203–214. Amer. Math. Soc., 1992.

[33] D. Saltman. Division algebras over p-adic curves. *J. Ramanujan Math. Soc.*, 12:25–47, 1997. see also the erratum [34] and survey [9].

[34] D. Saltman. Correction to division algebras over p-adic curves. *J. Ramanujan Math. Soc.*, 13:125–129, 1998.

[35] D. Saltman. Cyclic algebras over p-adic curves. *J. Algebra*, 314:817–843, 2007.

[36] V. Suresh. Bounding the symbol length in the Galois cohomology of function fields of p-adic curves. *Comment. Math. Helv.*, 85(2):337–346, 2010.

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